Lecture 6. Intersecting boundary strata.

(i) Fiber product along gluing morphisms

(ii) Normal bundle of $\overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$

(iii) Excess intersection formula.

Reference:

Graber, Pandharipande, Constructions of nontautological classes on moduli spaces of curves, 2003

§1. Overview

Recall Γ = stable graph of (g,n).

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)}$$

$$\mathcal{M}^\Gamma = \{ (C, p_1 \cdots p_n) \mid \text{dual graph} \cong \Gamma \} \subset \overline{\mathcal{M}}_{g,n}$$

$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \longrightarrow \overline{\mathcal{M}}_{g,n}$: gluing map where the image is $\overline{\mathcal{M}^\Gamma}$.

Goal Given two stable graphs Γ_1, Γ_2 with g, n, what is

$$\xi_{\Gamma_1 *} \mathbb{1} \cup \xi_{\Gamma_2 *} \mathbb{1} \quad \in H^*(\overline{\mathcal{M}}_{g,n}) \, ?$$

PD of the fundamental class of $\overline{\mathcal{M}}_{\Gamma_1}$

Q) cohomological deg of $\xi_{\Gamma_1 *} \mathbb{1} \stackrel{?}{=} \# E(\Gamma_1)$

To answer this question, we need 3 ingredients

(i) Fiber product

$$\begin{tikzcd} ?? \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{\Gamma_2} \arrow[d, "\xi_{\Gamma_2}"] \\ \overline{\mathcal{M}}_{\Gamma_1} \arrow[r, "\xi_{\Gamma_1}"'] & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

← fiber product of DM stacks.

deformation theory

(ii) Normal bundle of ξ_Γ

lci pullback.

(iii) Excess intersection formula

§2. Fiber product of two boundary strata

Let's start from set theoretic intersections in $\overline{M}_{g,n}$ ← coarse moduli space.

<u>Def</u> Let Γ, Γ' : stable graphs of (g,n). A morphism $\varphi : \Gamma' \to \Gamma$ consists of data

$$\varphi_V : V(\Gamma') \longrightarrow V(\Gamma), \quad \varphi_H : H(\Gamma) \longrightarrow H(\Gamma') \quad \text{s.t.}$$

(i) φ_H is injective

(ii) $\varphi_H : E(\Gamma) \longrightarrow E(\Gamma')$

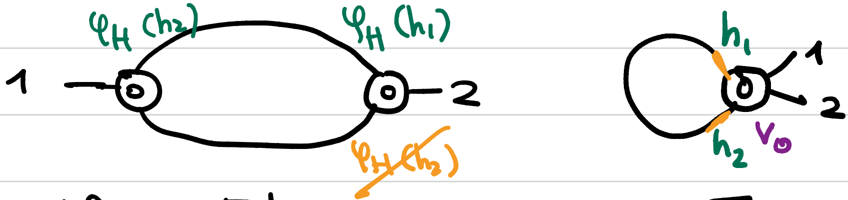
(iii) $\varphi_H : L(\Gamma) \longrightarrow L(\Gamma')$. preserving orders.

(iv) φ_V is surjective & compatible with φ_H :
let $v_\Gamma(h) \in V(\Gamma)$ be the vertex where h is attached.
Then

$$\varphi_V(v_{\Gamma'}(\varphi_H(h))) = v_\Gamma(h) \qquad \forall h \in H(\Gamma)$$

(v) We can obtain Γ' by gluing Γ_{v_0} into $v_0 \in V(\Gamma)$ for all v_0.

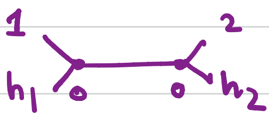
Example $(g,n) = (1,2)$



$$\varphi : \Gamma' \longrightarrow \Gamma$$

$$= \quad \varphi_V : V(\Gamma') \twoheadrightarrow V(\Gamma)$$

$$\varphi_H : H(\Gamma') \hookleftarrow H(\Gamma)$$

Glue 

at the vertex v_0 :

<u>Check</u> Given a morphism $\varphi : \Gamma' \to \Gamma$, $\exists$ gluing map

$$\xi_\varphi : \overline{\mathcal{M}}_{\Gamma'} \longrightarrow \overline{\mathcal{M}}_{\Gamma} .$$

Lemma A curve $(C, p_1 \cdots p_n) \in \overline{M}_{g,n}$ with the dual graph Γ' lies in $\overline{M}^{\Gamma}$ $\Leftrightarrow$ $\exists$ morphism $\Gamma' \to \Gamma$.
In particular,

$$\overline{M}^{\Gamma} = \bigcup_{\Gamma' \to \Gamma} M^{\Gamma'}.$$

Cor $\overline{M}^{\Gamma_1} \cap \overline{M}^{\Gamma_2} = \bigcup_{\substack{\Gamma' \\ \Gamma' \to \Gamma_1,\ \Gamma' \to \Gamma_2}} M^{\Gamma'}$ (set theoretic)

Now we move onto the fiber diagram

$$\begin{tikzcd}
F_{\Gamma_1 \Gamma_2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{\Gamma_2} \arrow[d, "\xi_{\Gamma_2}"] \\
\overline{M}_{\Gamma_1} \arrow[r, "\xi_{\Gamma_1}"] & \overline{M}_{g,n}
\end{tikzcd}$$

Def A (Γ_1, Γ_2)-structure on Γ is a tuple

$$(\Gamma, \varphi_1, \varphi_2) = (\Gamma, \varphi_1 : \Gamma \to \Gamma_1, \varphi_2 : \Gamma \to \Gamma_2).$$

We say two (Γ_1, Γ_2)-structures

$(\Gamma, \varphi_1, \varphi_2)$ and $(\Gamma', \varphi_1', \varphi_2')$

are isomorphic if $\exists$ isomorphism $\Gamma \xrightarrow{f} \Gamma'$ s.t

$$\begin{tikzcd}
 & \Gamma_1 & \\
\Gamma \arrow[ur, "\varphi_1"] \arrow[rr, "f", "\sim"'] \arrow[dr, "\varphi_2"'] & \circlearrowleft & \Gamma' \arrow[ul, "\varphi_1'"'] \arrow[dl, "\varphi_2'"] \\
 & \Gamma_2 &
\end{tikzcd}$$

We say $(\Gamma, \varphi_1, \varphi_2)$ is generic if

$$E(\Gamma) = \varphi_{1,E}(E(\Gamma_1)) \cup \varphi_{2,E}(E(\Gamma_2)).$$

$\hookrightarrow \# E(\Gamma) \leq \# E(\Gamma_1) + \# E(\Gamma_2)$

Let's denote

$$\mathcal{G}_{\Gamma_1\Gamma_2} = \{\text{generic } (\Gamma_1, \Gamma_2)\text{-structures}\}/\sim$$

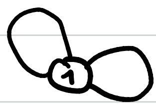
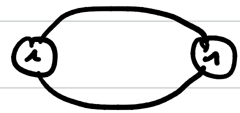
<u>Thm</u> (Graber-Pandharipande, '03)

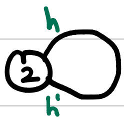
$$\begin{tikzcd}
\coprod_{\Gamma \in \mathcal{G}_{\Gamma_1\Gamma_2}} \overline{\mathcal{M}}_\Gamma \simeq F_{\Gamma_1\Gamma_2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{\Gamma_2} \arrow[d, "\xi_{\Gamma_2}"] \\
\overline{\mathcal{M}}_{\Gamma_1} \arrow[r, "\xi_{\Gamma_1}"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

You can find the proof in [GP.03].

Example $\Gamma_1 = \Gamma_2 =$  of (3.2).

Q) $G_{\Gamma_1,\Gamma_2} = \{(\Gamma, \varphi_1, \varphi_2)\}/\sim$??

Possible $\Gamma =$  ,  ,  .

When $\Gamma =$  , $\mathrm{Aut}\,\Gamma = \mathbb{Z}/2\mathbb{Z} = \{id, \varphi\}$

$\varphi : \Gamma \to \Gamma$, $\varphi_H(h) = h'$, $\varphi_H(h') = h$

$(\Gamma, id, id) \simeq (\Gamma, \varphi, \varphi)$

$$\begin{tikzcd} & \Gamma_1 & \\ \Gamma \arrow[ur, "id"] \arrow[dr, "id"'] \arrow[rr, "\varphi", "\sim"'] & & \Gamma \arrow[ul, "\varphi"'] \arrow[dl, "\varphi"] \\ & \Gamma_2 & \end{tikzcd}$$

$\not\simeq$

$(\Gamma, id, \varphi) \simeq (\Gamma, \varphi, id)$

$$\begin{tikzcd} & \Gamma_1 & \\ \Gamma \arrow[ur, "id"] \arrow[dr, "\varphi"'] \arrow[rr, "\varphi", "\sim"'] & & \Gamma \arrow[ul, "\varphi"'] \arrow[dl, "id"] \\ & \Gamma_2 & \end{tikzcd}$$

§3. Normal bundle to Ξ_Γ.

Fact.

$$T_{[C]}\overline{\mathcal{M}}_g = \mathrm{Hom}\left(\mathrm{Spec}\, k[\varepsilon]/\varepsilon^2, (\overline{\mathcal{M}}_g, [C])\right)$$

$$\cong \mathrm{Ext}^1_C(\Omega_C, \mathcal{O}_C)$$

Check When C: smooth, $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \cong H^1(C, T_C)$ and

$\dim H^1(C, T_C) = 3g-3$ if $g \geq 1$.

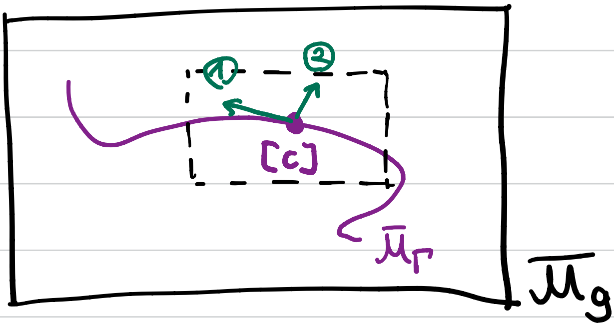
(Hint: RRoch).

By local-global spectral seq, the deform. space fits into:

$$0 \to \underbrace{H^1(\mathcal{H}om(\Omega_C, \mathcal{O}_C))}_{①} \to \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \to \underbrace{H^0(\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C))}_{②} \to 0 \qquad (*)$$

① locally trivial deformation : fixing nodes and varying complex structure of each Irred. components.

② deforming nodes : smoothing out nodes



Cohomological computation shows that (*) is a fiber of an exact sequence of a vector bundle on $\overline{\mathcal{M}}_\Gamma$:

$$0 \to T\overline{\mathcal{M}}_\Gamma \to \xi_\Gamma^* T\overline{\mathcal{M}}_g \to N_{\overline{\mathcal{M}}_\Gamma/\overline{\mathcal{M}}_g} \to 0$$

and

$$N_{\overline{\mathcal{M}}_\Gamma/\overline{\mathcal{M}}_g} \simeq \bigoplus_{(h,h')\in E(\Gamma)} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee .$$

(formula is the same for $\overline{\mathcal{M}}_{g,n}$).

$\mathbb{L}_h$ = cotangent line bundle at h

§4. Excess intersection formula.

Def Let $V = L_1 \oplus \cdots \oplus L_r$ (line bdls) be a split rank r vector bundle on X. Then the Euler class of V is defined by

$$e(V) = \prod_{i=1}^{r} c_1(L_i) \quad \in H^{2r}(X).$$

Lemma Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \ni \alpha \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

where X, X', Y, Y' : smooth, proper DM stacks

Then

$$g^* f_* \alpha = f'_* \left(e(E) \cup g'^* \alpha \right) \qquad (*)$$

where

$$E = (f')^* N_{Y'/Y} / N_{X'/X}. \qquad \text{"excess bundle"}$$

Eg $X \overset{i}{\hookrightarrow} Y$ closed embedding of smooth, proper varieties

$$\Rightarrow \quad i^* i_* \mathbb{1} = e(N_{X/Y}).$$

Check (LHS) & (RHS) of (*) have same cohomological degree.

We saw that for $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \longrightarrow \overline{\mathcal{M}}_{g,n}$,

$$\mathcal{N}_{\xi_\Gamma} \cong \bigoplus_{(h,h')\in E(\Gamma)} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee \quad \leftarrow \text{rank} = \# E(\Gamma).$$

Let's go back to the cartesian diagram :

$$\begin{tikzcd}
\coprod\limits_{\substack{(\Gamma,\varphi_1,\varphi_2)\\ \in \mathcal{G}_{\Gamma_1,\Gamma_2}}} \overline{\mathcal{M}}_\Gamma \arrow[r, "\xi_{\varphi_2}"] \arrow[d, "\xi_{\varphi_1}"'] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{\Gamma_2} \arrow[d, "\xi_{\Gamma_2}"] \\
\overline{\mathcal{M}}_{\Gamma_1} \arrow[r, "\xi_{\Gamma_1}"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

Prop We have

$$\xi_{\Gamma_1}^* \xi_{\Gamma_2 *} \mathbb{1} = \sum_{\substack{(\Gamma,\varphi_1,\varphi_2)\\ \in \mathcal{G}_{\Gamma_1,\Gamma_2}}} \xi_{\varphi_1 *}\left(e(E_\Gamma)\right)$$

where $E_\Gamma = \bigoplus_{\{h,h'\} \in \varphi_{1,E}(E(\Gamma_1)) \cap \varphi_{2,E}(E(\Gamma_2))} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee$: excess bundle on $\overline{\mathcal{M}}_\Gamma$.

I will leave the proof as an exercise. (Hint: use the excess intersection formula).

Cor $$\xi_{\Gamma_1 *} \mathbb{1} \cup \xi_{\Gamma_2 *} \mathbb{1} = \sum_{(\Gamma, \varphi_1, \varphi_2) \in \mathcal{G}_{\Gamma_1 \Gamma_2}} \xi_{\Gamma *}(e(E_\Gamma))$$

Pf) (LHS) $= \xi_{\Gamma_1 *}(\xi_{\Gamma_1}^* \xi_{\Gamma_2 *} \mathbb{1})$ $\quad\because$ projection formula

$= $ (RHS) $\quad\because \xi_{\Gamma *} = \xi_{\Gamma_1 *} \xi_{\varphi_1 *}$ + Proposition $\boxed{}$